Indian Statistical Institute, Bangalore
M. Math.
Second Year, First Semester
Operator Theory

Mid-Semestral Examination — Date : Sept. 29, 2010
Maximum marks: 100 — Time: 3 hours

In the following the field for vector spaces and algebras is taken to be the field of complex numbers and σ denotes spectrum.

1. Obtain the spectral decomposition for following matrices, that is, write them as unitary conjugates of diagonal matrices and also write them as linear combinations of projections.

$$M = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, N = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

[20]

2. Let X, Y be Banach spaces. Show that $K(X,Y)$, the space of all compact operators from X to Y is a closed subspace of the space $B(X,Y)$ of all bounded operators from X to Y. [15]

3. Let $\mathcal{F}$ be the algebra of all matrices of the form:

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

with complex numbers a, b. Show that $\mathcal{F}$ with

$$\| \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \| = |a| + |b|$$

is a commutative Banach algebra (You must verify all the axioms). Compute the spectrum of this Banach algebra. [15]

4. Let $\mathcal{A}$ be a unital Banach algebra. Consider a, b in $\mathcal{A}$.

   (i) Show that if $(1-ba)$ is invertible then so is $(1-ab)$. (Hint: If $c = (1-ba)^{-1}$, then $(1-ab)^{-1} = 1 + acb$).

   (ii) Show that $\sigma(ab) \bigcup \{0\} = \sigma(ba) \bigcup \{0\}$. [15]

5. Let $\mathcal{E} = C[0,1]$ be the Banach algebra of complex valued continuous functions on $[0,1]$. Let I be the ideal, $I = \{f : f \in \mathcal{E}, f(0) = f(1) = 0\}$. Show that the quotient space $\mathcal{E}/I$ is isomorphic to $\mathbb{C}^2$. [15]

[P.T.O.]

6. Let $\mathcal{A}$ be a unital commutative Banach algebra. Define the Gelfand map for $\mathcal{A}$ and show that the Gelfand map is a contractive homomorphism. [20]

7. Consider the set up of question 4. Suppose $\lambda.1 = ab - ba$, where λ is a scalar, show that $\lambda = 0$. (Hint: If $\lambda \neq 0$, arrive at a contradiction with 4(ii) by considering $\sigma(ab)$ and $\sigma(ba)$.) [10]